HEEGAARD-FLOER HOMOLOGY AND A FAMILY OF BRIESKORN SPHERES

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Abstract. We compute the Heegaard-Floer homology for the family $\Sigma(2,3,6n+1)$ of Brieskorn spheres using the algorithm given in [OS].

0. Introduction

Ozsváth and Szabó have given a combinatorial description for Heegaard-Floer homology in [OS]. Using this algorithm we compute Heegaard-Floer homology of $-\Sigma(2,3,6n+1)$:

Theorem 1. $HF^+(-\Sigma(2,3,6n+1)) = T_0^+ \oplus \mathbb{Z}^n_{(0)}$.

This family of homology spheres can be obtained by doing $-1/n$ surgery on right handed trefoil knot. This family and some more were considered in [FS] where their instanton Floer homology is calculated. For the Heegaard-Floer homology computations for the family $\Sigma(2,2n+1,4n+3)$ see [Ru]. Another combinatorial description for calculating HF^+ has been given by Némethi in [Ne].

1. Remarks on the algorithm

Consider a negative definite weighted graph G with at most one bad vertex (i.e., $d(v) > -v\cdot v$ for at most one vertex v where $d(v)$ denotes the number of edges containing the vertex v.) Let $X(G)$ denote the plumbed disk bundle over S^2 and $Y(G)$ its boundary. In [OS] the subset of $\mathbb{H}^+(-Y(G))$ of $\mathrm{Map}(Char(G), T_0^+)$ of finitely supported maps satisfying $U^{m+n}\cdot f(K+2PD[v]) = U^m \cdot f(K)$ whenever $min\{m, m+n\} \geq 0$ and $K\cdot v + v\cdot v = 2n$ is considered and the following is shown:

Theorem 2. [OS] *For such a graph G, for each $Spin^c$ structure t over $-Y(G)$,*

$$HF^+(-Y(G),t) \cong \mathbb{H}^+(G,t).$$

We will be working with homology spheres, so we suppress the $Spin^c$ structure from the notation. In the computations instead of working with elements of $\mathbb{H}^+(G)$, [OS] works with elements of $K^+(G)$, which is the set of equivalence classes of elements in $\mathbb{Z}^{\geq 0} \times Char(G)$, with the equivalence relation defined by

$$(m,K) \sim (m+n, K+2PD[v])$$

where v is a vertex in G with $K\cdot v + v\cdot v = 2n$ and $min(m, m+n) \geq 0$. Equivalence class of (m,K) will be denoted by $U^m \otimes K$.

For an equivalence class $U^m \otimes K$, define its U-depth as the smallest number l so that (l, K') is a representative of $U^m \otimes K$ for some vector K'. $K^+(G)$ is determined by elements of U-depth 0 and the U action on $K^+(G)$, which follows from

Proposition 3. *(Prop 3.2 in [OS]) For an equivalence class $U^m \otimes K$ of U-depth 0, there is a unique representative $(0,K)$ satisfying*

$$v_i\cdot v_i + 2 \leq K\cdot v_i \leq -v_i\cdot v_i \text{ for each } i. \tag{1}$$

Conversely if a vector K satisfies (1), then K has U-depth 0 if and only if K supports a good full path (in this case K will be called a basic vector).

In the above, full path stands for a path of vectors $K_1, K_2, ..., K_n$ in $Char(G)$ with K_1 satisfying (1) obtained by adding $2PD[v_i]$ if $K_i \cdot v_j + v_j \cdot v_j = 0$ for some j, until $-K_n$ satisfies (1) or $K_n \cdot v_j + v_j \cdot v_j > 0$ for some j. It is called good if $-K_n$ satisfies (1).

In the proof of [OS], Proposition 3.2, it is shown that given a characteristic vector M, the final vector of any full path is identical, hence we observe the following useful

Remark 1. Observe that if a vector supports a good full path, then all full paths are good, hence finding one bad full path means the initial vector is not basic. Secondly observe that bad full paths are hereditary, hence if a vector for a subgraph has a bad full path, then so does the containing vector and graph.

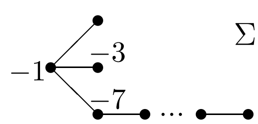
This will reduce the number of vectors that we need to check if they support a good full path or not. We will express the vectors $K \in Char(G)$ as sequences $(K \cdot v_i)$.

Lemma 4. *For the linear graph A_s with s vertices and each weight -2, there are no good full paths starting at vectors K satisfying (1) with $K \cdot v_i = 2$ for more than one i.*
Proof. We will use induction on s, observing that we can use hereditary property of bad full paths. (1) implies $K \cdot v_i \in \{0, 2\}$. For $s = 2$, $(2, 2)$ for A_2 has a bad full path obtained by adding $2PD(v_1)$. For $s > 2$, observe that if $K \cdot v_i = K \cdot v_j = 2$ for some $i \neq j$, then K is equivalent to a vector containing $(2, 2)$ as shown below, hence has a bad full path.

$$(*, 2, 0, 0, ..., 0, 0, 2, *') \sim (*'', -2, 2, 0, ..., 0, 0, 2, *') \sim (*'', -2, 0, ..., -2, 2, 2, *'). \square$$

2. The family $\Sigma(2, 3, 6n + 1)$

Consider the family of Brieskorn spheres $Y(n) = \Sigma(2, 3, 6n+1)$. The negative definite plumbing graph defining $Y(n)$ is a tree with weights -1 on central node, $-2, -3, -7$ on adjacent nodes and a -2 chain of length $n - 1$ starting at -7 as follows:



$\Sigma(2, 3, 6n + 1)$

The Heegaard-Floer homology of the first member of this family, $HF^+(-\Sigma(2, 3, 7))$, has been studied in [OS], [Ru].

Lemma 5. *For arbitrary n, the basic vectors for $Y(n) = \Sigma(2, 3, 6n + 1)$ are*

$$\begin{aligned}
K_1 &= (1, 0, -1, -5, 0, 0, 0, ..., 0)\\
K_2 &= (1, 0, -1, -3, 0, 0, 0, ..., 0)\\
K_3 &= (1, 0, -1, -5, 2, 0, 0, ..., 0)\\
K_4 &= (1, 0, -1, -5, 0, 2, 0, ..., 0)\\
K_5 &= (1, 0, -1, -5, 0, 0, 2, ..., 0)\\
&\vdots\\
K_{n+1} &= (1, 0, -1, -5, 0, 0, 0, ..., 2)
\end{aligned}$$

Proof. Clearly for each j, K_j satisfies (1). Next we need to see that among all characteristic vectors satisfying (1) these are the only ones supporting good full paths. For $n = 1$ this is done in [OS], and we verified it by computer.

By remark 1, for $n > 1$ first 4 entries of a basic vector has to coincide with one of $(1, 0 - 1, -3)$, $(1, 0, -1, -5)$ which were computed in [OS] for $n = 1$. Other entries are either 0 or 2. Moreover lemma 4 implies that for basic vectors K for $Y(n)$ there can be at most one vertex with $K \cdot v_i = 2$.
Claim. $(1, 0, -1, -3, *)$ has a bad path if $*$ has a non-zero entry.

Proof of claim. As in lemma 4, we can find a vector $(1, 0, -1, -3, 2, *')$ equivalent to K. But $(1, 0, -1, -3, 2)$ has a bad path obtained by addind $2PD(v_i)$ in the order $i = 1, 2, 1, 3, 1, 2, 1, 5, 4, 1, 2, 1$ and bad paths are hereditary. □
Therefore the only basic vector with initial segment $(1, 0, -1, -3)$ is $(1, 0, -1, -3, 0, ..., 0)$.

Next we need to show that $K_1, ..., K_{n+1}$ support good paths. This we do by explicitly giving the paths. First, $1, 2, 1, 3, 1, 2, 1$ is a good full path for both K_1 and K_2. For others, the path starts the same, but continues as:

$$\begin{array}{rl} 5,\ \ 6,\ \ 7, ..., n+3 & \text{for } K_3 \\ 6,5,\ \ 7,6,\ \ 8,7, ..., n+3, n+2 & \text{for } K_4 \\ 7,6,5,\ \ 8,7,6,\ \ 9,8,7, ..., n+3, n+2, n+1 & \text{for } K_5 \\ \vdots & \\ n+3, n+2, n+1, ..., 5 & \text{for } K_{n+1} \end{array}$$

This finishes the proof of the lemma. □

For each K_i, when we compute the renormalized lengths $\frac{K \cdot K + |G|}{4}$, each time we get 0. Next we investigate relationships between U powers of K_i.

Lemma 6. *$U \otimes K_i \sim U \otimes K_j \sim L = (-3, 2, 5, 1, 0, 0, ..., 0)$ for $1 \leq i, j \leq n + 1$*
Proof. For K_1, the sequence $1, 1, 2, 1$ leads to L.
For $i > 1$, the path from K_i leading to L is of the following form:

$$1, 1, 2, 1, 2, 3, 1,\ \ A_{n,i},\ \ B_n$$

where B_n is $1, 2, 3, 1, 4, 1, 2, 1$ followed by $5, 6, ..., n + 3$ followed by 1 and $A_{n,i}$ is of the following form:

$$C_{n-i+1}, C_{n-i+2}, ..., C_0.$$

In the above, C_k denotes the sequence $1, 2, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1,\ \ 5, 6, 4+k$ if $k > 0$ and empty path if $k = 0$. As an example, for $n = 4$, the path from K_2 to L is given by

$$\begin{aligned} 1, 1, 2, 1, 2, 3, 1, A_{4,2}, B_4 &= 1, 1, 2, 1, 2, 3, 1, C_3, C_2, C_1, B_4 \\ &= 1, 1, 2, 1, 2, 3, 1, \\ &\quad\ 1, 2, 3, 1, 4, 1, 1, 2, 1, 2, 3, 1, \\ &\quad\ 1, 2, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \\ &\quad\ 1, 2, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 6, \\ &\quad\ 1, 2, 3, 1, 4, 1, 2, 1, 5, 6, 7, 1 \end{aligned}$$

It is straightforward to check that these paths end at L. Next one observes that as in [AD], throughout these paths the U-depth stays between 0 and 1, hence we get $U \otimes K_i \sim L$ as announced. □

Now we know that $K^+(G)$ consists of $U^0 \otimes K_1, U^0 \otimes K_2$ and $U^m \otimes K_1$ for $m > 0$. For any f in $\mathbb{H}^+(G)$, $f(K_1) \in T_0^+$ determines the images $\tilde{f}(m, K_1)$ of the induced map $\tilde{f} : \mathbb{Z} \times Char(G) \to T_0^+$. The remaining values $f(K_i)$ for $i > 1$ are also determined up to addition of an element of $\mathbb{Z}_{(0)}$. This finishes the proof of the theorem. □

Acknowledgements. I would like to thank Selman Akbulut for advice, support and encouragement. Also I wish to thank John McCarthy and Yildiray Ozan for discussions and Dina Shnaider for encouragement.

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